Can we identify the relative price between consumption and investment?

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Abstract

This paper considers various AK models to investigate inference about the relative price between consumption and investment using NIPA data. We find, that depending on the model used, we can legitimately generate different time series for this price. If we successfully construct a falling price of investment, the model implies an inadmissibly low share of consumption in output. If we use an admissible share of consumption we generate investment prices which increase over time, contrary to the intuition generated by the price of equipment goods.

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1 Introduction

A key fact to have emerged from U.S National Income and Product Accounts (NIPA) data is that the price of equipment investment goods has fallen dramatically in the past 30 years relative to the price of consumption non-durable and services. Changes in this relative price are important because of its role in the decision to postpone consumption, thereby impacting on capital accumulation, the growth rate of output, and future welfare. Accordingly, the literature has devoted significant attention to this relative price.[1] In particular Greenwood, Hercovitz, and Krusell (1997) develop a model of capital embodied technological progress in equipment goods, where relative prices are determined by sectorial differences in the rate of technological progress.[2]

Equipment prices, however, are but one component of the broad intertemporal price relevant for the determination of aggregate savings and capital accumulation. This broad price is not just a measurable quantity but its relevance depends on the model one has in mind. Felbermayr and Licandro (2005), building on the framework proposed by Rebelo (1991), show that a two-sector AK model is consistent with several important characteristics observed from NIPA data. In particular that of a falling relative equipment price and real investment growth exceeding consumption growth.[3]

This paper uses a number of alternative AK models as a convenient tool to extract predictions for the relevant relative prices. Each model is used as a filter that takes quantity data as inputs to predict prices. The emphasis is therefore not on testing the models but rather on evaluating the inference induced by them. Very different outcomes for these predicted prices can be obtained, depending on the model as-

[1] See, for example, Chari, Kehoe and McGrattan (2001), Fisher (2003), and Whelan (2003).

[2] They briefly discuss issues related to this paper but they do not address our main question. In their model relative prices are *assumed* rather than inferred.

[3] Fatas (2000) and McGrattan (1998) also show that AK models match other features of the data.

sumptions and on the exact quantity data used to generate them. This suggests that inferences regarding the intertemporal trade-off, based on prices computed directly from the data, can be misleading.

We first consider a one-sector AK model that produces output using a linear production function, where output can be divided into consumption and savings. Savings are in turn transformed into investment via another linear production function. Both production functions are subject to specific technology shocks. The model is solved analytically and NIPA data is used to obtain a unique time series of the two shocks, such that the model exactly fits the data. These shocks are then used to construct the single relative price in the model. If quantity data for consumption is used an increasing price of investment relative to consumption is obtained. If investment quantity data is used, a relative price that falls across time is obtained. This relative price is the inverse of the technology shock from the savings technology. Both approaches are legitimate. However even if investment quantity data is used to generate a falling investment price, this comes at the cost of an inadmissibly low real consumption share.

Given that the relevant economic prices depend on the model, different models will yield different results. This is examined by considering a two-sector AK model studied by Felbermayr and Licandro (2005). In this model one sector produces consumption goods using a concave technology, and the other sector produces investment goods using a linear technology. Since investment goods are then used in both sectors, the model predicts that consumption will become more expensive as capital accumulates in the economy. Thus the declining price of investment, is not fundamentally a result of technological progress, but rather a result of the asymmetric sectorial impact of capital accumulation. This model is also biased towards this relative price outcome because it only permits the use of investment quantity data to construct the shocks needed to predict prices. However both of these bias fade away as the production technology for consumption becomes linear, such that the model converges to the one-sector model, where the relative price is determined by exogenous technology changes in the investment sector.

The final exercise in this paper considers a three-sector model in order to decompose investment into equipment and structures. Here the relative prices observed directly in the data can differ from those implied by the model, such that a different view of the intertemporal significance of equipment prices can be obtained. In addition, this disaggregated model is used to highlight one final issue: in general, aggregate output is not uniquely defined, such that the inference crucially depends on how one chooses to define it. This is a problem related to the construction of indices for real aggregates in NIPA data.

The next section revisits the one-sector AK model and discusses the main problems in obtaining the predictions of such models. Section 3 considers a two-sector growth model and Section 4 examines the three-sector AK model to study the impact of different capital goods on the broad relative price. An appendix details the index algebra used to construct consumption, investment and output variables from NIPA data. Section 5 concludes.

2 A One-sector AK Model

2.1 Model Outline

Consider the following one-sector growth model.[4] Utility of the representative agent is maximized subject to a resource utilization constraint where aggregate output is

[4] For this economy, the solution to the planner's problem coincides with the market solution, because all markets are assumed to be perfectly competitive.

divided between consumption and savings, $y_t = A_t k_t = c_t + s_t$. Production is of the AK form, where A_t is an intratemporal technology shock. There is also an intertemporal technology that transforms current savings into investment, $I_t = \theta_t s_t$, which is summarized by the shock θ_t. An increase in θ_t constitutes an increase in the efficiency of the intertemporal technology. Capital accumulation obeys the following law of motion:

$$k_{t+1} = (1 - \delta)\, k_t + I_t, \tag{1}$$

where capital depreciates at rate $0 < \delta < 1$.

The problem of the planner is to choose an investment path to maximize the sum of the present value of expected utility flows

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t),$$

subject to the resource utilization constraint

$$c_t = A_t k_t - \frac{I_t}{\theta_t} = A_t k_t - \left(\frac{k_{t+1} - (1-\delta) k_t}{\theta_t} \right). \tag{2}$$

Given a discount factor $0 < \beta < 1$, and solving with respect to k_{t+1}, one can obtain the Euler equation of this economy:

$$u'(c_t) = \theta_t \beta E_t \left\{ u'(c_{t+1}) \left[A_{t+1} + \frac{1-\delta}{\theta_{t+1}} \right] \right\}. \tag{3}$$

With logarithmic utility the dynamic programming problem can be solved analytically. The policy function for this model is then given by:

$$k_{t+1} = \beta \left[\theta_t A_t + (1 - \delta) \right] k_t. \tag{4}$$

Using the resource constraint (2), the policy function (4) and the law of motion for capital (1), it is straightforward to obtain the consumption share and the growth rates for consumption, output and investment:

$$\frac{c_t}{y_t} = (1-\beta)\left[1 + \frac{(1-\delta)}{A_t\theta_t}\right] \tag{5}$$

$$\frac{c_{t+1}}{c_t} = \frac{\beta\theta_t}{\theta_{t+1}}\left[A_{t+1}\theta_{t+1} + (1-\delta)\right] \tag{6}$$

$$\frac{y_{t+1}}{y_t} = \frac{A_{t+1}}{A_t}\beta\left[A_t\theta_t + (1-\delta)\right] \tag{7}$$

$$\frac{I_{t+1}}{I_t} = \beta\left[A_t\theta_t + (1-\delta)\right]\left[\frac{\beta A_{t+1}\theta_{t+1} - (1-\beta)(1-\delta)}{\beta A_t\theta_t - (1-\beta)(1-\delta)}\right]. \tag{8}$$

2.2 Recovering the Shocks

The task now is to obtain the shocks A_t and θ_t. Here, since the aim is to infer relative prices, quantity data is used to enable the model to predict the prices. It is important to be clear from the outset that the model data and the actual data are treated in exactly the same way, through employing a Fisher chain-aggregation approach.[5] The construction of chained-type quantity indexes, for both the actual and model data, avoids the well-known substitution bias inherent in fixed-based quantity indexes.[6] In terms of the model, the time series for θ can be interpreted as the relative price between investment and consumption (P_I/P_C). This follows from equation (2) where along the resource constraint the trade-off between consumption and investment is given by

$$\frac{\partial c}{\partial I} = -\frac{1}{\theta}.$$

[5]Appendix 6.1 contains details of the construction of the variables.

[6]See for example, Whelan (2002, 2003) and Fisher and Shell (1998).

However given the setup of the model there are two alternative ways to construct the shocks. Consumption share data can be used to obtain $\{\theta_t, z_t\}$ or instead investment growth data can be used to predict the implied relative price.[7]

First suppose that θ_t and A_t are constructed using data series for output and consumption. Using equations (5) to (7), it is straightforward to obtain an expression for the growth of θ_t which depends on consumption growth and the share of consumption:[8]

$$\frac{\theta_t}{\theta_{t-1}} = \frac{\frac{c_t}{y_t}}{\frac{c_t}{y_t} - (1-\beta)} \frac{\beta (1-\delta)}{\frac{c_t}{c_{t-1}}} \tag{9}$$

Starting from an initial value of $\theta = 1$, the series for θ_t is constructed by iterating on equation (9), using the data on c_t and y_t described below in Appendix 6.1. Figure 1 depicts the first 30 observations of the series $\{\theta_t\}$ for the parameter values $\beta = 0.94$ and $\delta = 0.1$, which shows that the computed series starts at 1 and falls geometrically from then on.[9] This outcome is very robust to variations in β and δ and data manipulations on how the quantity index for the real consumption share is constructed.[10] However the finding that the price of investment goods is growing faster than the price of consumption goods is at odds with the data depicted in Figure 2. Here the corresponding relative price measured in the data shows a clear fall in the investment price relative to consumption, from the normalized value of 1 in 1950 to around 0.45 in 2004. Therefore, the relative price prediction of this model is misleading.

[7] Nevertheless, as Ingram, Kocherlakota, and Savin (1994) show, with two shocks the model can never be rejected.

[8] Given the series obtained for θ_t, the time series for the shock A_t is obtained using equation (5).

[9] The A_t shock grows exponentially. The product of the two, $A_t\theta_t$, is stationary.

[10] As discussed in Appendix 6.2.2, there are alternative ways to construct the real (c/y) share using quantity indices. However this does not qualitatively affect the series for $\{\theta_t\}$ that is recovered.

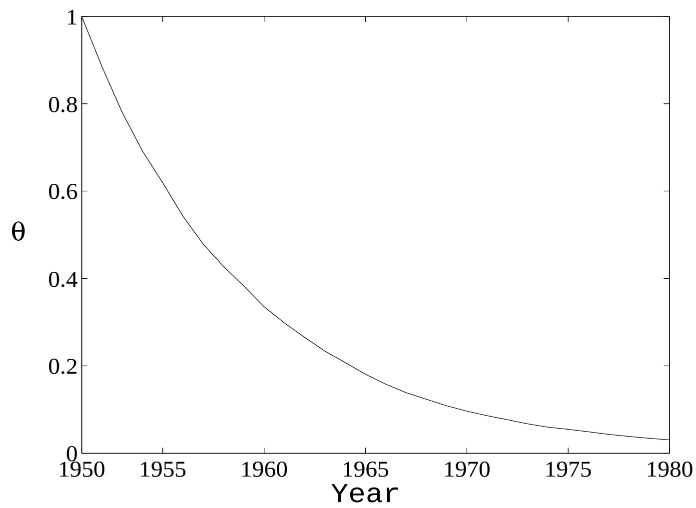

Figure 1: θ shock obtained using consumption data

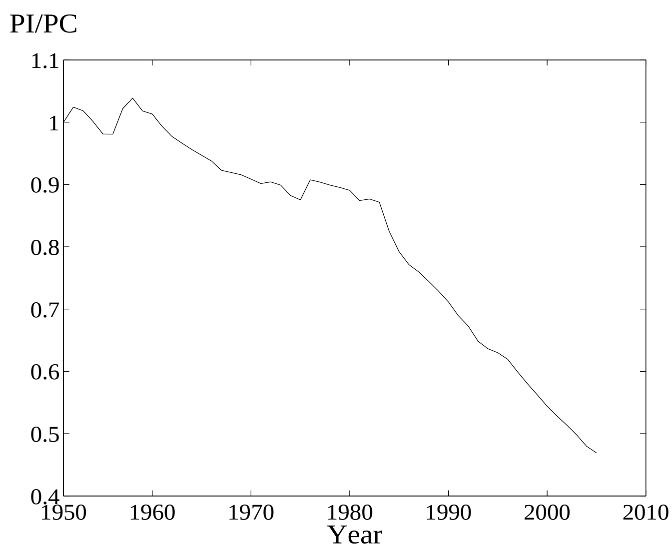

Figure 2: Relative price (PI/PC) from actual data

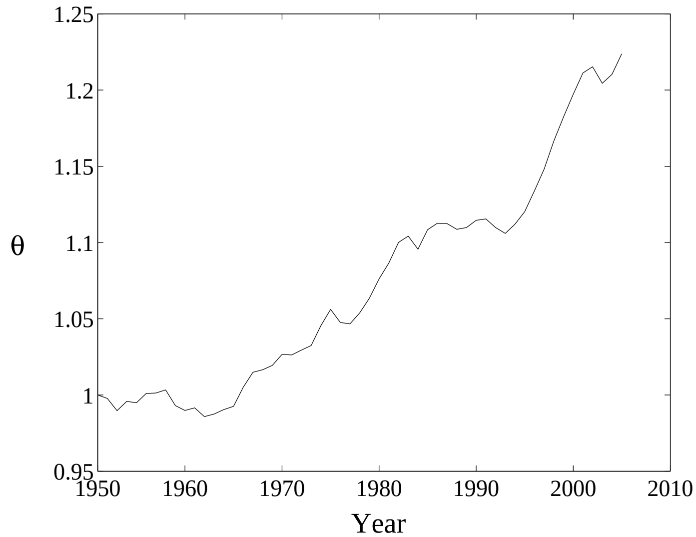

Figure 3: θ shock obtained using investment data

Alternatively, now suppose that the shocks A_t and θ_t are constructed using data series for investment. Here the growth rate of investment can be used to compute the product $A_t\theta_t$.[11] After obtaining $A_t\theta_t$, A_{t+1}/A_t is computed using the growth rate for output, equation (7). Once the time series for A_t is computed,the time series for θ_t can be recovered, which is depicted in Figure 3. By inspection, the θ_t series obtained using investment data is very different compared to the series constructed using consumption data. In addition since θ_t is increasing this implies that the time series for A_t has a downward trend, since their product is stationary.

However, obtaining the shocks this way has one important strong implication: the consumption share implied by $A_t\theta_t$, constructed from

$$\frac{c_t}{y_t} = (1-\beta) + (1-\beta)\,\frac{(1-\delta)}{A_t\theta_t}$$

has a mean of 0.3173 with no apparent trend. This value of 32% is interesting.

[11] I_{t+1}/I_t is stationary so $A_t\theta_t$ is initialized at the value implied by the mean of I_{t+1}/I_t and iterated on equation (8).

Imposing a constant consumption share χ on equation (9),

$$\frac{\theta_t}{\theta_{t-1}} = \frac{\frac{c_t}{y_t}}{\frac{c_t}{y_t} - (1-\beta)} \frac{\beta(1-\delta)}{\frac{c_t}{c_{t-1}}} = \frac{\chi}{\chi - (1-\beta)} \frac{\beta(1-\delta)}{\frac{c_t}{c_{t-1}}}$$

stationarity of θ_t is achieved around $\chi = 0.33$. At $\chi = 0.34$ the θ_t series clearly falls.[12] When θ_t is constructed using consumption share data:

$$\frac{\theta_t}{\theta_{t-1}} = \frac{\frac{c_t}{y_t}}{\frac{c_t}{y_t} - (1-\beta)} \frac{\beta(1-\delta)}{\frac{c_t}{c_{t-1}}} = \frac{y_{t-1}}{y_t} \frac{\frac{c_{t-1}}{y_{t-1}}}{\frac{c_t}{y_t} - (1-\beta)} \beta(1-\delta)$$

then given a stable consumption share, what determines the evolution of θ_t is dependent upon the growth of output. However there is one caveat: how close the real consumption share is to the value of $(1-\beta)$ crucially matters. The real consumption share index used to construct θ_t was initialized at the same value as its nominal share 0.74. This share then falls over time getting closer to $(1-\beta)$ but not close enough to overcome the impact of consumption and output growth in this sample.[13] However when investment data are used, the model implies a value of $\chi = 0.32$ which explains why an increasing θ series was obtained.

Hence the initial value of the real share of consumption is important for the θ series implied by the model. Using constructed consumption share data, a price of investment that is increasing relative to consumption is obtained, whereas if investment quantity data is used a falling relative price is recovered, but at the cost of an inadmissibly low consumption share. Note however that while the real share of

[12] The nominal consumption share is very stable, with a mean of 0.744 and a standard deviation of the log equal to 0.0193. For the implicit series the mean is 0.3173 and the standard deviation of the log is 0.0479. Neither series has a significant trend. In addition, if the value of β is increased, the threshold value of stationarity falls. For example if $\beta = 0.96$, this threshold value is around $\chi = 0.25$, making this a robust problem.

[13] We also used a real share computed using a Fisher index, without any significant change in θ. See details in Appendix 6.2.2.

consumption is clearly defined in the model, it is not clearly defined in the data. Therefore it is essentially only common sense that suggests that a value of 0.32 is inadmissible. Therefore the analysis suggests that virtually any relative price can be generated through the appropriate manipulation of real c/y.

So far we conclude that relative prices, constructed from quantity data and the model, cannot be safely predicted. Given the model, exactly how it is matched with the data may yield vastly conflicting outcomes. However, as the next section highlights, it is important to emphasize that outcomes are not only data dependent but also model dependent as well.[14]

3 A Two-sector AK Model

3.1 Model Outline

An alternative AK growth model is explored in Felbermayr and Licandro (2005). In this model one sector produces consumption goods using concave technology and the other sector produces investment goods according to a linear production function. Specifically the two production technologies for consumption and investment goods are given by:

$$C_t = A_t\left(k_t^c\right)^{\alpha}$$

$$I_t = A_t \theta_t (k_t^i)$$

where $0 < \alpha < 1$, $(A_t,\, \theta_t)$ are technology shocks and the superscripts c and i denote the respective consumption and investment sectors. Full employment implies that

[14] Indeed, Ejarque and Reis (2004) develop a model of endogenous growth where the relative price between consumption and investment is indeterminate, and thus meaningless.

$k_t = k_t^c + k_t^i$ and capital accumulation obeys the law of motion given by equation (1). In this model aggregate output is no longer uniquely defined. One can choose to define it by the identity $Y_t = C_t - \frac{\partial C}{\partial I} I_t$, or it can be defined as a chained-aggregated index, an issue we return to later in Section 4.

As before, the problem of the planner is to choose an investment path to maximize the sum of the present value of expected utility flows, subject to a different resource utilization constraint given by:

$$C_t = A_t \left[\frac{k_t A_t \theta_t - k_{t+1} + (1-\delta) k_t}{A_t \theta_t} \right]^{\alpha}. \tag{10}$$

Given a discount factor $0 < \beta < 1$, and assuming that utility is the logarithm of consumption, it is straightforward to show that the policy function for this model is still given by equation (4).

Using the resource constraint, the policy function and the law of motion for capital, one can reproduce the same growth rate for investment as given by equation (8). However the growth rate for consumption is now given by:

$$\frac{C_{t+1}}{C_t} = \frac{A_{t+1}}{A_t} \left[\frac{A_t \theta_t}{A_{t+1} \theta_{t+1}} \right]^{\alpha} \left[\frac{(1-\beta) A_{t+1} \theta_{t+1} + (1-\delta)(1-\beta)}{(1-\beta) A_t \theta_t + (1-\delta)(1-\beta)} \right]^{\alpha} \beta \left[(1-\delta) + A_t \theta_t \right].$$

In this model the only way to extract the shocks θ_t and A_t is to use equation (8) to construct the product $A_t \theta_t$ and then use other expressions to extract the A_t shock. This limits the quantity data that can be used to recover the shocks and therefore biases the inference towards obtaining an increasing series for θ_t and a declining price of investment. However, from the conclusions of the previous section, this may still not lead to a satisfactory outcome.

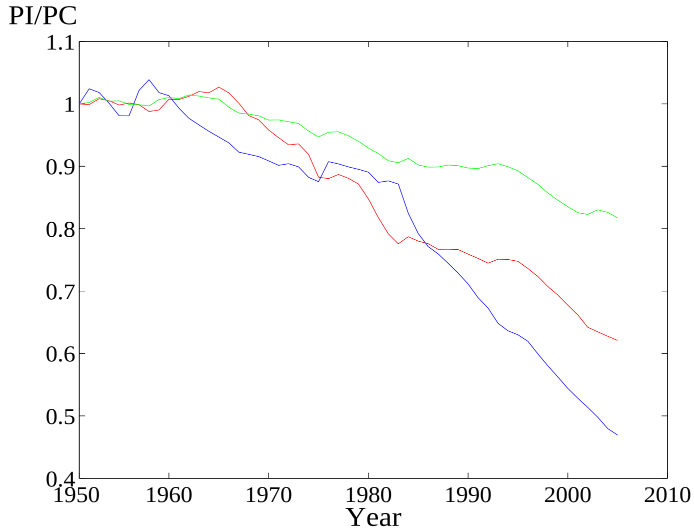

Figure 4: PI/PC

3.2 Model Predictions

If $\alpha = 1$ this model is *identical* to the previous one-sector model with the results based on investment data reported above. If $\alpha \approx 1$, θ is an increasing series, but as α is reduced this is reversed. Therefore variations in α (concavity) directly affect the series for θ that is obtained.[15]. However, given the assumption $0 < \alpha < 1$, $1/\theta$ in this model is no longer the relative price. The trade-off between consumption and investment along the resource constraint (10) is now given by:

$$\frac{\partial C}{\partial I} = -\frac{1}{A_t\theta_t}\alpha\frac{C_t}{k_t^c} = -\frac{C_t}{I_t}\frac{\alpha}{(1-\beta)}\left[\frac{\beta A_t\theta_t - (1-\delta)(1-\beta)}{A_t\theta_t + (1-\delta)}\right]$$

where quantity data for consumption and investment is used, together with the constructed shocks $A_t\theta_t$, to obtain the investment price relative to consumption. This relative price recovered is illustrated by the bottom line in Figure 4 which, for comparison, also depicts the relative prices obtained from the actual data (middle line)

[15] With $\beta = 0.94$ and $\delta = 0.1$ stationarity for θ is achieved when $\alpha \approx 0.8$

and the previous one-sector model using investment data (top line). As can be seen, this price is decreasing across time, which is easy to understand since $A_t\theta_t$ is stationary from that data and real investment grows more than real consumption.[16] Even though the trajectory of θ_t depends on α, it is important to stress that since only equation (8) can be used to construct the shocks, the relative price obtained is decreasing over time irrespective of the value of α. While the price *level* depends on the value of α, its *path*, normalized by the initial value, is independent of α. This helps explain the crucial difference between the two models. In the previous model the relative price obtained ($1/\theta$) depended only upon changes in technology. Here, however, capital accumulation rather than exogenous technology changes, is the key factor behind the predicted relative price. If α is close to one, the θ_t series is increasing and this delivers the falling relative price of investment. If α falls below 0.8, then given the other parameter values, concavity takes over the task of generating this falling relative price. However now the technology shock series θ_t decreases over time, implying that the price of capital falls despite the technological *regression*. Therefore, given a little concavity, this model inevitably generates a falling relative price, since as the stock of capital grows, then by assumption, investment becomes more productive relative to consumption.[17] Furthermore since the use of the data is also limited, it biases the inference in the same direction.

The consumption share implied by $A_t\theta_t$, constructed from

$$\frac{C_t}{Y_t} = \frac{(1-\beta)\left[A_t\theta_t + (1-\delta)\right]}{(1-\beta)(1-\delta)(1-\alpha) + A_t\theta_t\left[1-\beta(1-\alpha)\right]}$$

is crucially affected by the value of α. Figure 5 depicts this implied consumption share

[16]The quantity indices for consumption and investment grow at 3.48% and 4.53% respectively.

[17]In this sense the one-sector model is more agnostic regarding relative prices. This is also in contrast to Greenwood, Hercowitz and Krusell (1997) model where relative prices are determined by the rate of technological progress between sectors.

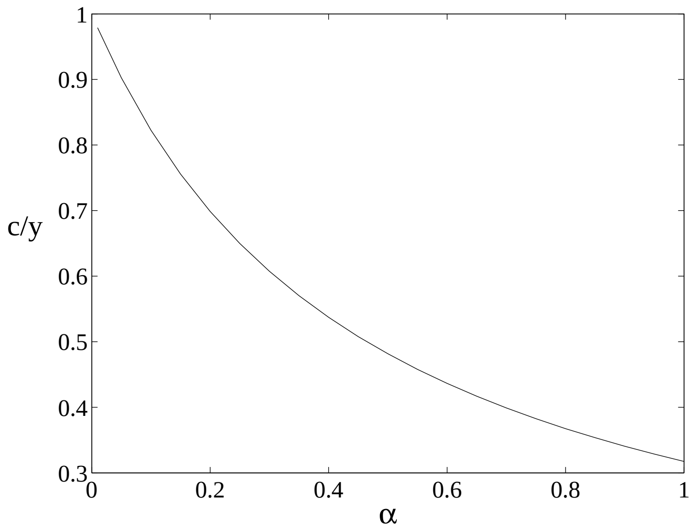

Figure 5: Implied real C/Y given the mean $(A_t\theta_t)$

for variations in α using the mean $(A_t\theta_t)$: the higher is α the lower is the implied consumption share.[18] Therefore, a falling price of investment relative to consumption can be obtained, for any consumption share greater than 0.3173.[19] However whether the consumption share is admissible or not depends on the value of α. The problem here is the implication that a low α is needed for a high consumption share, but this also implies a falling series for θ_t, or a technological regression in the investment sector. Thus this model requires an empirical measure of the degree of returns to scale in the consumption goods sector, given that linearity in investment is a maintained assumption of endogenous growth.[20]

We have discussed the main issues in using these models to infer relative prices. First, different quantity data, consumption share data or investment growth data,

[18]The product $A_t\theta_t$ is obtained using equation (8) which is a stationary series & is independent of α.

[19]Clearly when $\alpha = 1$ both models imply the same real consumption share.

[20]If labor is explicit, the utility function matters. If $C_t = A_t\left(k_t^c\right)^\alpha L_t^\phi$, and $U = \log(C_t) - \frac{\tau\phi}{1+\psi}L_t^{1+\psi}$, and L only enters the production of consumption goods, then the exponent of k is unchanged at α. But if utility is given by $U = log\left(A_t\left(k_t^c\right)^\alpha L_t^\phi - \frac{\tau\phi}{1+\psi}L_t^{1+\psi}\right)$, then a different power of k in the production function is obtained.

imply different predictions for relative prices. This is fundamentally due to differences in the implied real consumption share. There is no way around this problem since the real consumption share is not observed and can only be constructed from nominal data using Fisher indices, which do not aggregate additively. Consequently there is no right or wrong outcome for the different series generated above. Second, the choice of model is also important, not least as it crucially conditions the quantity data that can be used.

One particular caveat of the models considered so far is that they do not distinguish between different types of capital and thus provides no guide to the aggregation of different components of investment. To address such issues, the next section develops a three-sector growth model to investigate how the disaggregation of capital influences our conclusions to date.

4 A Three-sector AK model

4.1 Model Outline

Consider the following three-sector version of the previous model, which now consists of a consumption sector and two investment sectors. Each investment sector employs only its sector-specific capital denoted k and h, to produce its respective output I or J. In the consumption sector both types of capital are combined under a Cobb-Douglas production function to produce a consumption good C. Specifically the production technologies for consumption and investment goods are given by:

$$C_t = A_t\left(k_t^c\right)^{\alpha}\left(h_t^c\right)^{\gamma}$$

$$I_t = \theta_t k_t^i$$

$$J_t = z_t h_t^j$$

where $\alpha > 0$, $\gamma > 0$ and $\alpha + \gamma < 1$; A_t, θ_t and z_t are technology shocks; and the superscripts c, i and j denote the respective sectors. Full employment implies that $k_t = k_t^c + k_t^i$, and $h_t = h_t^c + h_t^j$. Sector specific investment and capital accumulation obey

$$k_{t+1} - (1-\delta_k)\, k_t = I_t = \theta_t\left[k_t - k_t^c\right] \tag{11}$$

$$h_{t+1} - (1-\delta_h)\, h_t = J_t = z_t\left[h_t - h_t^c\right] \tag{12}$$

where (δ_k, δ_h) are the sector-specific depreciation rates.

The problem of the planner is to choose an investment path to maximize the sum of the present value of expected utility flows subject to the resource constraint

$$C_t = A_t\left(\frac{\psi_t^k k_t - k_{t+1}}{\theta_t}\right)^{\alpha}\left(\frac{\psi_t^h h_t - h_{t+1}}{z_t}\right)^{\gamma} \tag{13}$$

where $\psi_t^k = (1-\delta_k)+\theta_t$, and $\psi_t^h = (1-\delta_h)+z_t$. With log utility, the Euler equations for the planner's problem are:

$$\begin{aligned}
\frac{1}{\psi_t^k k_t - k_{t+1}} &= \beta E_t\left\{\frac{1}{\psi_{t+1}^k k_{t+1} - k_{t+2}}\psi_{t+1}^k\right\} \\
\frac{1}{\psi_t^h h_t - h_{t+1}} &= \beta E_t\left\{\frac{1}{\psi_{t+1}^h h_{t+1} - h_{t+2}}\psi_{t+1}^h\right\}
\end{aligned}$$

and the policy functions of this model are given by:

$$k_{t+1} = \beta\psi_t^k k_t \tag{14}$$

$$h_{t+1} = \beta\psi_t^h h_t. \tag{15}$$

Note that this solution, and both Euler equations, are independent of consumption good technology (α, γ).[21]

4.2 Relative Price Predictions

Using the resource constraint (13), the policy functions $(14-15)$ and the laws of motion for capital $(11-12)$, one can obtain the growth rates for investment and consumption to construct the shocks:

$$\begin{aligned}
\frac{I_{t+1}}{I_t} &= \frac{\Omega_{t+1}^k}{\Omega_t^k}\beta\psi_t^k \\
\frac{J_{t+1}}{J_t} &= \frac{\Omega_{t+1}^h}{\Omega_t^h}\beta\psi_t^h \\
\frac{c_{t+1}}{c_t} &= \frac{A_{t+1}}{A_t}\left(\frac{\theta_t}{\theta_{t+1}}\psi_{t+1}^k\beta\right)^{\alpha}\left(\frac{z_t}{z_{t+1}}\psi_{t+1}^h\beta\right)^{\gamma}
\end{aligned}$$

where $\Omega_t^k = \left[\beta\theta_t - (1-\delta_k)(1-\beta)\right]k_t$ and $\Omega_t^h = \left[\beta z_t - (1-\delta_h)(1-\beta)\right]h_t$. Initial values for (z_t, θ_t) are set using the sample average of the investment growth equations, as in both cases the growth rates are stationary in the data. The initial value of (A_t) is just set at 1 where the consumption growth equation is used to recover the A_t shock. The θ_t shock obtained displays large fluctuations around a slight positive trend, whereas the z_t shock increases until mid sample and then falls, being hard to

[21] It is easy to show that, as long as θ_t and z_t do not grow or decline too fast, the transversality conditions are satisfied for any value of α and γ. See Appendix 6.3 for further details.

discern a trend.[22]

For this three-sector model, it is useful to solve for the prices that generate a decentralized equilibrium. If all three sectors are characterized by perfect competition, then given the rental rates for sector specific capital, r_t^h and r_t^k, profit maximization in each sector implies the following first order conditions:

$$\begin{aligned}\alpha A_t \left(k_t^c\right)^{\alpha-1}\left(h_t^c\right)^{\gamma} &= r_t^k \\ \gamma A_t \left(k_t^c\right)^{\alpha}\left(h_t^c\right)^{\gamma-1} &= r_t^h\end{aligned}$$

and for the investment sectors, $p_t^i\theta_t = r_t^k$, and $p_t^j z_t = r_t^h$, where the price in sector I (J) is denoted p^i (p^j) with $p_t^c = 1$ for all periods.[23] After some straightforward substitutions one arrives at the relative price between capital goods

$$\frac{p_t^i}{p_t^j} = \frac{\alpha\psi_t^h}{\gamma\psi_t^k}\frac{\Omega_t^k}{\Omega_t^h}\frac{J_t}{I_t}$$

which depends on the quantity indices I and J and on the parameters of the model. The narrow relative price of each capital good with respect to consumption is

$$\begin{aligned}\frac{p_t^i}{p_t^c} &= p_t^i = \frac{C_t}{I_t}\frac{\alpha}{(1-\beta)}\frac{\Omega_t^k}{\psi_t^k} \\ \frac{p_t^j}{p_t^c} &= p_t^j = \frac{C_t}{J_t}\frac{\gamma}{(1-\beta)}\frac{\Omega_t^h}{\psi_t^h}\end{aligned}$$

and the relative price of broad capital is then computed using a Fisher index.[24]

To use the model to infer the shocks, we need values for the five parameters $(\beta, \alpha, \gamma, \delta_k, \delta_h)$, and use Whelan (2003) to obtain the benchmark. The depreciation

[22] Recall that (z_t, θ_t) are independent of the values of (α, γ).

[23] Since only relative prices matter for equilibrium, consumption is the numeraire.

[24] See Appendix 6.3.1.

rates are chosen as $\delta_k = 0.13$, and $\delta_h = 0.03$, as these are the typical values used to construct the NIPA capital stocks for durable equipment and structures, and the production technology parameters are set at $\alpha = 0.145$ and $\gamma = 0.165$. The discount factor is chosen at $\beta = 0.94$. It is important to note that these parameter values are used for aggregate models which differ from the model used here. While it is natural to use the benchmark for the physical depreciation rates, the equivalence of (α, γ) in this model with that of aggregate models does not necessarily follow.[25]

Figure 6 depicts the relative prices (relative to consumption prices) found in the data and Figure 7 depicts the relative prices predicted by the model. In each figure, the middle line is the price of broad capital, which is a Fisher index that includes equipment and structures. The upper line is the price of structures and the lower line is the price of equipment. It is transparent to see that the three-sector growth model robustly produces the pattern for relative prices found in the data, which is equivalent to the results generated by the previous models when investment data was used. However an additional observation emerges: the prices generated by the model are similar but not identical to the Fisher price indices that where constructed using the data. While this difference may not seem remarkable, and is not fundamental for the main point of this paper, it is still the case that the price of broad capital index falls to about 0.5 in the actual data, whereas the same index implied by the model only falls to about 0.8, a significant difference. The model is a filter that takes quantity data as inputs to predict prices, and shows that inference based on prices computed directly from the data can be misleading.

[25] α and γ are parameters of the production function of nondurable goods and services, not of aggregate output.

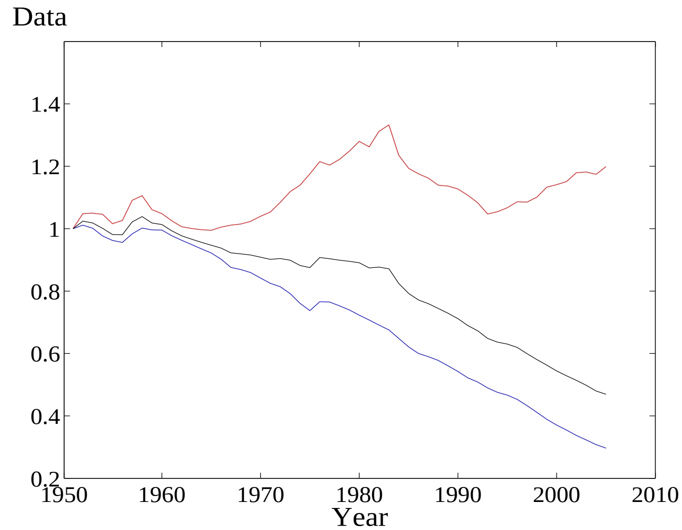

Figure 6: Relative Prices obtained from Actual Data

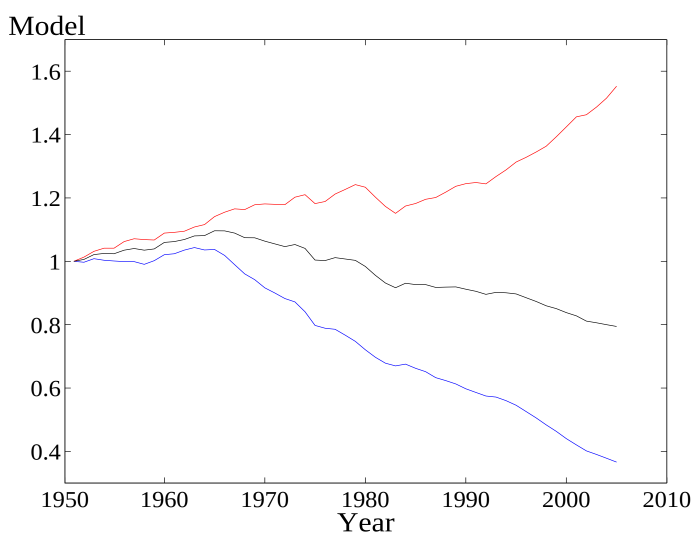

Figure 7: Relative Prices obtained from Model

4.3 Real Consumption Share

If we define output as $Y = C - \frac{\partial C}{\partial I} I - \frac{\partial C}{\partial J} J$ this yields a consumption share given by:

$$\frac{C}{Y} = \frac{(1-\beta)\psi_t^k \psi_t^h}{(1-\beta)\psi_t^k \psi_t^h + \alpha \Omega_t^k \psi_t^h + \gamma \Omega_t^h \psi_t^k}.$$

Since neither the shocks θ_t and z_t nor the parameters Ω_t^k, Ω_t^h, ψ_t^k and ψ_t^h, are affected by $(\alpha,\, \gamma)$ it is possible to see how changes in the production technology parameters affect the implied consumption share when output is defined in this way. Using the mean of $A_t\theta_t$ and $A_t z_t$, the implied real consumption share is calculated, for different values of $0 < \alpha + \gamma < 1$. Similar to the two-sector model and as highlighted by Table 1, the lower the concavity, the lower is the implied real consumption share. The consumption share is also stationary in this construction given values for α and γ. For example if $\alpha = 0.145$ and $\gamma = 0.165$, then a real consumption share series is obtained that starts from a value of 0.6517 in 1950 and ends at a value of 0.6605 by 2004 with a mean of 0.6505 and a standard deviation equal to 0.0161.[26] Of course under the assigned parameter values for α and γ, this is still a significant degree of concavity by comparison with the linear technology for investment goods.[27]

However an alternative way to define aggregate output is to construct a real expenditure series using chained-aggregated data. In this case a completely different series for the real consumption share is obtained and this highlights the inherent difficulties in trying to measure this share when aggregate output is not uniquely defined in the model. The series constructed using a Fisher index for output declines dramatically. Given an initial value of 1 in 1950, the series declines to 0.4738 by

[26] The series $A_t\theta_t$ and $A_t z_t$ are both stationary.

[27] See, for example, Hornstein and Pracschick (1997) and Huffman and Wynne (1999), for evidence that this difference in concavity may not be found in the data. For instance, Huffman and Wynne estimate the power of capital in the production function of consumption goods at 0.41, while for investment goods this value is 0.34.

		α		
		0.05	0.25	0.45
	0.05	0.8478	0.5797	0.4406
ψ	0.25	0.7405	0.5274	0.4097
	0.45	0.6577	0.4838	0.3828

Table 1: Implied real consumption share for variations in α and γ

2004. Therefore we can conclude that since relative price shifts are important, the construction of the real consumption share based on direct addition will lead to a significant difference in the results obtained. This arises because the first approach fails to account for relative price movements within the two types of investment.[28]

5 Concluding Remarks

In this paper three AK models are matched with the data to evaluate their quantity and price implications. We find that the relative price predicted by the models depends crucially on what part of the data is used. Using a constructed consumption share, a price of investment that is increasing relative to consumption is obtained, or using investment quantity data a falling price of investment relative to consumption is obtained, but at the cost of an inadmissibly low consumption share. There is no right or wrong way to do this exercise. However one can bias the inference towards a declining relative price of investment, either through the choice of model or through the choice of quantity data to use. For instance the two and three sector AK models considered, unambiguously predict a falling relative price because they are limited in their use of data. Even so, since in this multi-sector environment the definition of output is not unique, different definitions of aggregate output yield different conclusions for the implied real consumption share. This is directly related to the index

[28]See Whelan (2002) for more on the additive problems when using chain-aggregated data.

number construction in NIPA data. The use of NIPA data to match these models must be done with care. What we show here, that even if this is the case, robust inference is not necessarily obtained, particularly with regard to the broad relative price between consumption and investment.

6 Appendices

6.1 Data

We use yearly National Income and Product Accounts (NIPA) data over the period $1950 - 2004$ obtained from the U.S. Department of Commerce, Bureau of Economic Analysis (BEA). From 1996 the BEA calculates all published real aggregates according to a chain-index formula.[29] We define C to be the consumption of nondurable goods and services, (columns 5 and 6 in the NIPA tables), I to be consumption of durable goods and investment in equipment and software, (columns 4 and 11), and J to be investment in structures (column 10). While a real series for J is directly available from the NIPA , our definitions of C and I each require the aggregation of two real series to produce a series not currently made available through the BEA. Given the additivity problems associated with chain-index data, as recommended by Whelan (2002) we construct real quantity indexes for C and I using a Fisher chain-aggregate formula (the square root of the product of a Paasche and a Laspeyres index) that replicates the procedure used by the BEA in producing the NIPA accounts.

The data extracted from the NIPA tables contains nominal quantities (m) and indices for prices (p^x) and real quantities (q^x). The price and real quantity indices are normalized to 100 in the year 2000. The indices and nominal quantities are related by the equation:

$$m_t = \frac{p_t^x}{100}\frac{q_t^x}{100} \times m_{2000}$$

Before proceeding, we renormalize the indices dividing by their initial value. This way, all indices will have value 1 in 1950, instead of 100 in 2000. This is useful because

[29] For information on the construction of NIPA accounts visit the BEA website at {www.bea.doc.gov/bea/dn1.htm} and see Whelan (2002). We use: “Real Gross Domestic Product, Quantity Indexes (Yearly)” , “Gross Domestic Product, Billions of Dollars (Yearly)”, and finally “Implicit Price Deflators, Index numbers (Yearly)”.

the Fisher indices constructed below also have initial value of 1.

We need a new price index and a new quantity index for consumption that aggregates nondurables (n) and services (s). The Fisher quantity index is:

$$F_t^c = \left[\frac{q_t^n p_t^n + q_t^s p_t^s}{q_{t-1}^n p_t^n + q_{t-1}^s p_t^s} \frac{q_t^n p_{t-1}^n + q_t^s p_{t-1}^s}{q_{t-1}^n p_{t-1}^n + q_{t-1}^s p_{t-1}^s} \right]^{\frac{1}{2}}$$

and then

$$Q_t^c = F_t^c \times Q_{t-1}^c$$

with initial value $Q_1^c = 1$. Inverting the time index on prices and quantities we get the price index P^c.[30] The price indices must be divided by the price index for consumption, so that they become comparable with the model.

[30] We can construct implicit price deflators, defined as ratios of the current dollar value series to the corresponding chained dollar value series. These are almost exactly the same as the price indices computed in the text and do not change the results.

6.2 One-sector AK model

6.2.1 Planner's Problem

Choosing k as the endogenous state variable with $(A,\,\theta)$ being the set of exogenous state values, the value function for this two-sector growth model is given by:

$$V(k_t, A_t, \theta_t) = \max_{k_{t+1}} \{U(c_t) + \beta E_t V(k_{t+1}, A_{t+1}, \theta_{t+1})\}$$

subject to the resource utilization constraint:

$$c_t = A_t k_t - \left[\frac{k_{t+1} - (1-\delta)k_t}{\theta_t}\right].$$

Letting λ denote the Lagrangian multiplier, the first-order conditions are:

$$U'(c_t) - \lambda_t = 0 \tag{A1}$$

$$\beta E_t V(k_{t+1}) - \frac{\lambda_t}{\theta_t} = 0 \tag{A2}$$

$$V(k_t) = \lambda_t \left[A_t + \frac{(1-\delta)}{\theta_t}\right] \tag{A3}$$

Forwarding (A3) and substituting this along with (A1) into (A2) yields the Euler equation (3.3).With $U = \log c_t$ the Euler equation becomes:

$$\frac{1}{c_t} = \theta_t \beta E_t \left\{ \frac{1}{c_{t+1}} \left[A_{t+1} + \frac{(1-\delta)}{\theta_{t+1}}\right]\right\} \tag{A4}$$

Guess that the policy function takes the form

$$k_{t+1} = \mu \theta_t \left[A_t + \frac{(1-\delta)}{\theta_t}\right] k_t$$

where μ is a constant to be determined. Using the resource constraint to eliminate c_t from (A4) and substituting this guess into the Euler equation gives $\mu = \beta$.

6.2.2 Data in the one-sector AK model

We use the equation

$$\frac{\theta_t}{\theta_{t-1}} = \frac{\frac{c_t}{y_t}}{\frac{c_t}{y_t} - (1-\beta)} \frac{\beta(1-\delta)}{\frac{c_t}{c_{t-1}}}$$

to generate the θ_t series, which we initialize at 1. But to do so, we need data on real shares of spending, and on real consumption growth. The consumption indices are constructed as described in Appendix 3.6.1 above and the output index aggregates all five components using a Fisher index.

We first construct a real share as the division of the quantity indices

$$\alpha_t^c = \frac{c_t}{y_t} = \frac{Q_t^c}{Q_t^y}$$

and multiply this equation by the initial nominal share. This index declines from the initial value[31] of 0.74 down to around 0.55 at the end of the sample.

For illustrative and robustness purposes we also construct an ad-hoc consumption real share by using the quantity indices, Q_t^c and Q_1^I, in the following way.

$$\begin{aligned}
\alpha_t^c &= \frac{c_t}{y_t} = \frac{c_t}{c_t + I_t} = \frac{Q_t^c}{Q_t^c + Q_t^I} \\
\alpha_{t+1}^c &= \frac{Q_{t+1}^c}{Q_{t+1}^c + Q_{t+1}^I} = \frac{(Q_{t+1}^c/Q_t^c)\alpha_t^c}{(Q_{t+1}^c/Q_t^c)\alpha_t^c + (Q_{t+1}^I/Q_t^I)(1-\alpha_t^c)}
\end{aligned}$$

where we initialize the real consumption share (α_0^c) to equal the nominal consumption initial share. The nominal share of consumption in output is stable around 0.74, but the real share of consumption, as constructed here, falls from 0.74 to around 0.64.

[31] Set at the same value as the initial nominal share.

However, this decline is not enough to change the outcome of a decreasing θ.

6.3 Three-sector AK Model

6.3.1 Planner's problem and prices

It is straightforward to write the Bellman equation of this problem. The value of entering the current period, V(S), is a function of the state vector ($S = [A, z, \theta, k, h]$), and is given by

$$V(S) = \max_{k',h'} \{\log(c_t) + \beta E_t V(S')\}$$

where primes denote future values. As before, it is also straightforward to derive the Euler equations and show that the policy functions we use in the text satisfy the Euler equations. Given these we can show that, under certain conditions analyzed above for the behavior of the shocks, we find a value function which is necessarily unique and we satisfy the transversality conditions (see below). This is enough to ensure we have the unique solution to our problem.

If all three sectors are characterized by perfect competition, then given the rental rates for sector specific capital, r_t^h and r_t^k, profits in each sector are given by

$$\begin{aligned}
\pi_t^c &= A_t (k_t^c)^\alpha (h_t^c)^\gamma - r_t^h h_t^c - r_t^k k_t^c \\
\pi_t^i &= p_t^i \theta_t k_t^i - r_t^k k_t^i \\
\pi_t^j &= p_t^j z_t h_t^j - r_t^h h_t^j
\end{aligned}$$

and profit maximization implies $p_t^i \theta_t = r_t^k$, $p_t^j z_t = r_t^h$, and in the consumption sector

$$\begin{aligned} \alpha A_t \left(k_t^c\right)^{\alpha-1} \left(h_t^c\right)^{\gamma} &= r_t^k \\ \gamma A_t \left(k_t^c\right)^{\alpha} \left(h_t^c\right)^{\gamma-1} &= r_t^h \end{aligned}$$

Now if we eliminate the rental rates we can find the relative prices:

$$\begin{aligned} \alpha \frac{A_t}{\theta_t} \left(k_t^c\right)^{\alpha-1} \left(h_t^c\right)^{\gamma} &= p_t^i \\ \gamma \frac{A_t}{z_t} \left(k_t^c\right)^{\alpha} \left(h_t^c\right)^{\gamma-1} &= p_t^j \end{aligned}$$

and now we make use of the policy functions and of the production functions, we obtain the price expressions we use in the main text.

An alternative way to compute a broad price of capital uses an index weighted by nominal expenditure weights:

$$\begin{aligned} \frac{\tilde{p}}{p_t^c} &= \tilde{p} = p_t^i \left[\frac{p_t^i I_t}{p_t^i I_t + p_t^j J_t}\right] + p_t^j \left[\frac{p_t^j J_t}{p_t^I I_t + p_t^j J_t}\right] \\ \tilde{p} &= \frac{C_t}{(1-\beta)} \left[\left(\frac{\alpha \Omega_t^k}{\psi_t^k I_t}\right) \left[\frac{\alpha \Omega_t^k \psi_t^h}{\alpha \Omega_t^k \psi_t^h + \gamma \Omega_t^h \psi_t^k}\right] + \left(\frac{\gamma \Omega_t^h}{\psi_t^h J_t}\right) \left[\frac{\gamma \Omega_t^h \psi_t^k}{\alpha \Omega_t^k \psi_t^h + \gamma \Omega_t^h \psi_t^k}\right]\right] \end{aligned}$$

This will yield similar results to the fisher index.

6.3.2 Transversality Condition

Utility is given by

$$
\begin{aligned}
u_t &= \log\left[A_t\left(\frac{\psi_t^k}{\theta_t}(1-\beta)\right)^{\alpha}\left(\frac{\psi_t^h}{z_t}(1-\beta)\right)^{\gamma}\right]+\log\left(k_t^{\alpha}h_t^{\gamma}\right) \\
u_{t+1} &= \log\left[A_{t+1}\left(\frac{\psi_{t+1}^k}{\theta_{t+1}}(1-\beta)\right)^{\alpha}\left(\frac{\psi_{t+1}^h}{z_{t+1}}(1-\beta)\right)^{\gamma}\right]+\log\left(k_{t+1}^{\alpha}h_{t+1}^{\gamma}\right) \\
\log\left(k_{t+1}^{\alpha}h_{t+1}^{\gamma}\right) &= \log\left(k_t^{\alpha}h_t^{\gamma}\right)+\log\left(\left(\beta\psi_t^k\right)^{\alpha}\left(\beta\psi_t^h\right)^{\gamma}\right) \\
\log\left(k_{t+2}^{\alpha}h_{t+2}^{\gamma}\right) &= \log\left(k_t^{\alpha}h_t^{\gamma}\right)+\log\left(\left(\beta\psi_t^k\beta\psi_{t+1}^k\right)^{\alpha}\left(\beta\psi_t^h\beta\psi_{t+1}^h\right)^{\gamma}\right)
\end{aligned}
$$

and we can write them as

$$
\begin{aligned}
u_t &= B_t+\log\left(k_t^{\alpha}h_t^{\gamma}\right) \\
u_{t+1} &= B_{t+1}+\log\left(k_t^{\alpha}h_t^{\gamma}\right)+\log\left(\left(\beta\psi_t^k\right)^{\alpha}\left(\beta\psi_t^h\right)^{\gamma}\right) \\
u_{t+2} &= B_{t+2}+\log\left(k_t^{\alpha}h_t^{\gamma}\right)+\log\left(\left(\beta\psi_t^k\beta\psi_{t+1}^k\right)^{\alpha}\left(\beta\psi_t^h\beta\psi_{t+1}^h\right)^{\gamma}\right)
\end{aligned}
$$

and now the present value of all this is given by

$$
V = u_t+\beta E_t\left(u_{t+1}\right)+\beta^2 E_t\left(u_{t+2}\right)+...
$$

and so we write

$$
\begin{aligned}
V &= \frac{1}{1-\beta}\log\left(k_t^{\alpha}h_t^{\gamma}\right)+E_t\sum_{j=0}^{\infty}\beta^j B_{t+j} \\
&\quad +E_t\sum_{j=1}^{\infty}\left[\beta^j\log\left(\left(\prod_{i=1}^{j}\psi_{t+i-1}^k\right)^{\alpha}\left(\prod_{i=1}^{j}\psi_{t+i-1}^h\right)^{\gamma}\right)\right]+(\alpha+\gamma)\log(\beta)\sum_{j=1}^{\infty}\beta^j j
\end{aligned}
$$

where the first and last terms are clearly finite irrespective of the values of (α,γ). The question is about the two middle terms. Take the first middle term, and assume some constant growth rates for the A shock, $A_{t+j} = A_t(1+g_a)^j$:

$$\begin{aligned}
W &= E_t \sum_{j=0}^{\infty} \beta^j B_{t+j} = E_t \sum_{j=0}^{\infty} \beta^j \log \left[A_{t+j} \left(\frac{\psi_{t+j}^k}{\theta_{t+j}} (1-\beta) \right)^{\alpha} \left(\frac{\psi_{t+j}^h}{z_{t+j}} (1-\beta) \right)^{\gamma} \right] \\
W &= \sum_{j=0}^{\infty} \beta^j \log \left[A_t (1+g_a)^j \left(\frac{\psi_{t+j}^k}{\theta_{t+j}} (1-\beta) \right)^{\alpha} \left(\frac{\psi_{t+j}^h}{z_{t+j}} (1-\beta) \right)^{\gamma} \right] \\
W &= \frac{\log(A_t) + (\alpha+\gamma)\log(1-\beta)}{1-\beta} + \frac{\beta \log(1+g_a)}{(1-\beta)^2} + W_1
\end{aligned}$$

The last term (W_1) is what worries us. We take a look at it now:

$$W_1 = \alpha \sum_{j=0}^{\infty} \beta^j \log \left(\frac{\psi_{t+j}^k}{\theta_{t+j}} \right) + \gamma \sum_{j=0}^{\infty} \beta^j \log \left(\frac{\psi_{t+j}^h}{z_{t+j}} \right)$$

Clearly, if θ_t and z_t tend to zero, we need these effects to be dominated by the discounting. If these shocks are stationary these terms are finite and we have no problem. If they have a positive growth rate again we have no problem as both fractions will tend to 1. Since we do not know the properties of the different shocks we will leave this discussion for now. Note however, that the value of $\alpha+\gamma$ is not an issue until now.

The second middle term is

$$Y = E_t \sum_{j=1}^{\infty} \beta^j \left[\alpha \log \left(\prod_{i=1}^{j} \psi_{t+i-1}^k \right) + \gamma \log \left(\prod_{i=1}^{j} \psi_{t+i-1}^h \right) \right]$$

and again it is easy to show that this expression is finite if the two shocks are stationary. We can show easily that this sum is finite if θ_t and z_t are constants. This expression is also finite if the two shocks tend to zero. If they grow, we cannot have them growing very fast.

Therefore, as long as θ_t and z_t do not grow or decline too fast, we will have a well defined problem and satisfy the transversality conditions.

References

[1] Chari, V.V., Kehoe, P., and H. McGrattan, 2001. The Poverty of Nations. Manuscript, Federal Reserve Bank of Minneapolis.

[2] Ejarque, J., and A. Reis, 2004. (Relative Price) Lessons from Taking an AK Model to the Data. Working Paper, University of Essex.

[3] Fatás, A., 2000. Endogenous Growth and Stochastic Trends. *Journal of Monetary Economics* 45, pp107-128.

[4] Felbermayr, G., and O. Licandro, 2005. The Under-Estimated Virtues of the Two-Sector AK Model. *Contributions to Macroeconomics* 5, Article 9.

[5] Fisher, F.M., and K. Shell, 1998. Economic Analysis of Production Price Indexes. New York: Cambridge University Press.

[6] Fisher, J., 2003. Technology Shocks Matter. Working Paper, Federal Reserve Bank of Chicago, pp2002-2014.

[7] Greenwood, J., Hercovitz, Z. and P. Krusell, 1997. Long-run Implications of Investment-specific Technological Change. *American Economic Review* 87, pp342-362.

[8] Hornstein, A., and J. Praschnik, 1997. Intermediate Inputs and Sectoral Comovement in the Business Cycle. *Journal of Monetary Economics* 40, pp573-595.

[9] Huffman, G.W., and M.A. Wynne, 1999. The Role of Intratemporal Adjustment Costs in a Multi-sector Economy. *Journal of Monetary Economics* 43, pp317-350.

[10] Ingram, B., Kocherlakota, N., and N. Savin, 1994. Explaining Business Cycles. A Multiple-shock Approach. *Journal of Monetary Economics* 34, pp415-428.

[11] Jones, C., 1995. Time Series Tests of Endogenous Growth Models. *Quarterly Journal of Economics* 110, pp495-525.

[12] McGrattan, E., 1998. A Defense of AK Growth Models. *Federal Reserve Bank of Minneapolis Quarterly Review* 22, pp13-27.

[13] Rebelo, S., 1991. Long Run Policy Analysis and Long Run Growth. *Journal of Political Economy* 99, pp500-521.

[14] Whelan, K., 2002. A Guide to the use of Chain Aggregated NIPA Data. *Review of Income and Wealth* 48, pp217-233.

[15] Whelan, K., 2003. A Two-sector Approach to Modelling U.S. NIPA Data. *Journal of Money Credit and Banking* 35, pp627-656.